The Cost of Multiple Representations:

Learning Number Symbols with Abstract and Concrete Representations

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Raw data, analysis scripts and experimental materials are available at the following location:

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Abstract

Parents are frequently advised to use number books to help their children learn the meaning of number words and symbols. How should these resources be designed to best support learning? Previous research has shown that number books typically include multiple concrete representations of number. However, a large body of mathematics education research has demonstrated that there may be costs, as well as benefits, to using both multiple representations and concrete representations when learning mathematical concepts. Here we used an artificial symbol learning paradigm to explore whether the use of abstract (arrays of dots) or multiple concrete (changing arrays of pictures) numerical representations resulted in better learning of novel numerical symbols by children. Across three experiments we found that children who learned the meaning of novel symbols by pairing them with numerosities represented by arrays of dots performed better on a subsequent symbolic comparison task than those who paired them with multiple concrete representations, or a mixture of abstract and multiple concrete representations. This advantage was not due to abstract representations being inherently superior to concrete representations, but instead to the use of *multiple* concrete representations. We conclude that the very common practice of using multiple concrete representations in children's number books may not be the most effective to support children's early number learning.

Educational Impact and Implications Statement

Parents and teachers often help children to learn the magnitude meanings of number words and symbols by reading number books with them. Here we investigated whether it is more effective to represent magnitudes in such books using multiple concrete representations (five fish, five pizzas, ten cars, ten sheep, etc.), single concrete representations (five fish, five fish, ten fish, ten fish, etc.) or abstract representations (five dots, five dots, ten dots, ten dots, etc.). Across three experiments we found that there was a cost to using multiple concrete representations compared to either single concrete representations or abstract representations. We conclude that number books designed to introduce children to the magnitude meanings of number words and symbols should consider using simple representations.

How do young children attach meaning to number words and symbols? This is a question that has drawn attention from researchers in developmental psychology and mathematics education alike. Despite considerable research interest there is still no consensus about what gives number words and symbols their meaning (the 'symbol grounding problem', Leibovich & Ansari, 2016) or how young children come to acquire number words. One consequence of this gap in knowledge is that we don't know how to develop the most effective resources to support preschool children's early number learning.

Even before children begin formal schooling there are wide individual differences in numerical skills (Ginsburg, Lee, & Boyd, 2008) and these differences predict later mathematical achievement (Duncan et al., 2007; Jordan, Kaplan, Ramineni, & Locuniak, 2009; Melhuish et al., 2008). Children who start school with poor numeracy skills are unlikely to catch-up with their peers (Jordan, Kaplan, Locuniak, & Ramineni, 2007). Therefore, in recent years there has been an increased focus on the development of children's mathematical skills in their early years. It has been suggested that one important influence on preschool numerical skills is the type and frequency of number-based activities that parents engage in with their children at home (LeFevre et al., 2009; Ramani, Rowe, Eason, & Leech, 2015; Skwarchuk, Sowinski, & LeFevre, 2014).

One of the most popular activities that parents do with their young children to improve number skills is reading number books (LeFevre et al., 2009). Parents typically receive encouragement to read to their children, and the notion that books support mathematical learning is not new (Hassinger-Das, Jordan, & Dyson, 2015; Jennings, Jennings, Richey, & Dixon-Krauss, 1992; Vandermaas-Peeler, Nelson, Bumpass, & Sassine, 2009; Wade & Moore, 2000). Indeed Peterson et al. (2014) found that training with number books was more effective than training with physical objects to support young children's

understanding of cardinality. However, at present we do not know how number books can be designed to most effectively help children learn the meaning of number words and symbols. In particular, there is little evidence regarding the types of numerosity representations that might be most beneficial to include in such books.

In this paper, we investigate the effects of using different types of numerosity representations when children learn number symbols. Drawing upon the extensive literature that has explored the use of abstract or concrete representations in mathematics learning more generally, we test the benefits and costs of using different types of representations to support children's learning of novel number symbols.

Learning from abstract and concrete representations

Decades of research in mathematics education has debated whether real-world or abstract representations are more effective when teaching abstract mathematical concepts. This debate is often characterized as a choice between 'abstract' or 'concrete' representations. However, researchers do not agree upon a definition of abstract and concrete (Sarama & Clements, 2009; Wilensky, 1991) and it is sensible to think of a continuum between completely abstract representations at one extreme and completely concrete representations at the other. Here we follow Fyfe, McNeil, Son, & Goldstone's (2014) approach: concrete representations are considered to be those that "connect with learners' prior knowledge, are grounded in perceptual and/or motor experiences, and have identifiable correspondences between their form and referents" (p. 1) while abstract representations are those that "eliminate extraneous perceptual properties, represent structure efficiently, and are more arbitrarily linked to their referents" (p. 1-2). For example, a (relatively) concrete representation of the fraction ½ might be (a picture of) half a pizza

whereas a (relatively) abstract representation might be a square with half of the area

shaded. When dealing with abstract mathematical concepts such as numbers, the third of

Fyfe et al's (2014) criteria is not straightforward to interpret. Arguably the correspondence

between representations and numerical concepts such as "half" is no less arbitrary for a

picture of half a pizza than for a half shaded square. Nevertheless these two representations

clearly differ in the amount of extraneous information provided and the connections with

learners' prior knowledge. Therefore it seems worthwhile to consider the distinction

between more abstract and more concrete representations in the context of number

concepts.

For many years it was generally accepted that concrete representations were

beneficial when introducing mathematical concepts to learners, particularly young children.

Piaget (1971) suggested that young children's cognitive abilities are not mature enough to

fully engage in abstract thinking and therefore that concrete representations are necessary

to aid their learning. Bruner (1966) went further and argued that all learners, not only young

children, benefit from being presented with new information in a concrete form before

being introduced to the abstract form. These theories resulted in a general acceptance that

children should learn about mathematical concepts through concrete representations.

Two main advantages to the use of concrete representations have been proposed.

First, they allow learners to activate real world knowledge to help them solve problems or

understand mathematical ideas (Kotovsky, Hayes, & Simon, 1985; Schliemann & Carraher,

2002). Second, concrete representations improve memory and understanding by giving the

learner an imagined action related to that mathematical concept (Glenberg, Gutierrez,

Levin, Japuntich, & Kaschak, 2004). For example, the idea of division might be introduced

with the example of sharing cookies between friends. Children can draw on their real world

experiences to understand the idea of sharing equally and, when asked to divide in future problems, they may be more likely to remember the procedure via the imagined action of sharing cookies. In addition to these cognitive advantages it has been suggested that concrete representations have a motivational benefit. For example, LeFevre & Dixon (1986) demonstrated that students prefer working with concrete representations. Students were presented with conflicting abstract instructions and a concrete representation and they predominately chose to follow the concrete representation.

Despite the early support for concrete representations, in recent years there has been a growing body of research suggesting that concrete representations may not be universally beneficial when learning mathematics. Several costs associated with the use of concrete representations have been highlighted (Brown, McNeil, & Glenberg, 2009). First, concrete representations contain extraneous details that can distract the learner from the relevant information (DeLoache, 2000; Uttal, O'Doherty, Newland, Hand, & DeLoache, 2009), a phenomenon that has been referred to as the 'seductive details effect' (Garner, Gillingham, & White, 1989; Harp & Mayer, 1998). Second, it has been shown that learning through concrete representations can constrain transfer of knowledge to other problems (Day, Motz, & Goldstone, 2015; Sloutsky, Kaminski, & Heckler, 2005). For example, introducing fractions with concrete examples of cutting a cake can constrain children's transfer of knowledge to other problems, particularly if they are asked to multiply or divide fractions, as it no longer makes sense to think about fractions as portions of cake.

In support of this view, a high-profile paper by Kaminski, Sloutsky, & Heckler (2008) suggested that learning from abstract representations results in a more sophisticated level of understanding of mathematical concepts than learning from concrete representations. Kaminski et al. (2008) compared groups of undergraduates who learned about an algebraic

concept – the group of order 3 – using abstract or concrete representations. Following a learning phase, participants' understanding was assessed using a multiple-choice test. Participants who learned the mathematical concept with an abstract representation performed better on the post-test than those who learned with a concrete representation. Kaminski et al. (2008) concluded that abstract representations are more effective than concrete representations when learning mathematical concepts. However, critics have argued that the questions in the post-test were more similar to the learning phase stimuli seen by the abstract learning group than those seen by the concrete learning group and that the two groups could have learned somewhat different concepts (group of order 3 vs. addition modulo 3) from the representations they were given (Jones, 2009a, 2009b; Mourrat, 2008; De Bock, Deprez, Van Dooren, Roelens, & Verschaffel, 2011).

Although these criticisms limit the conclusions which can be drawn from Kaminski et al.'s (2008) study, other researchers have also suggested that there can be advantages to using abstract representations (Koedinger, Alibali, & Nathan, 2008; McNeil, Uttal, Jarvin, & Sternberg, 2009). As implied above, unlike concrete representations, abstract representations eliminate extraneous information. Consequently they may focus attention on the to-be-learned information more effectively and the knowledge gained is more generalizable for transfer to other problems (Son, Smith & Goldstone, 2008, Uttal et al., 2009). On the other hand, there may also be costs to using abstract representations. For example Koedinger & Nathan (2004) found that students relied on formal solution methods, which were less likely to lead to a correct answer, when using abstract representations compared with situated story problems. For story problems, which provided greater context, students made more use of informal methods based on the problem situation and they were consequently more accurate. It has also been suggested that knowledge gained

from abstract representations is less likely to be flexibly applied to new problems (McNeil & Alibali, 2005) and the use of abstract representations can lead to logical errors when learners fail to understand the problem situation (Carraher & Schliemann, 1985).

Given the potential advantages and disadvantages of both concrete and abstract representations, a third approach, known as *concreteness fading*, has been proposed (Bruner, 1966; Fyfe, et al., 2014; Goldstone & Son, 2005). The aim of the concreteness fading technique is to combine the advantages of both types of representation. In this approach, learning begins with concrete representations, which allow the learner to access real-world concepts to help understand the key idea, and then gradually fades to more abstract representations, which have the advantage that the information learnt can be transferred to other problems. Bruner described an expanded version of this with three forms: an enactive form which is a physical concrete model of the concept; an iconic form which is a graphic pictorial model of the concept and a symbolic form which is an abstract model of the concept. For example, the quantity 'two' could first be represented by two physical items, e.g. apples, then these could be gradually replaced by a picture of two dots. These representations differ in the amount of extraneous information included and the connections to learners' prior experiences. Several researchers have adopted the concreteness fading approach, finding positive effects (see Fyfe, et al., 2014 for review). For example, Goldstone & Son (2005) investigated students' learning and transfer of a scientific principle by presenting them with a concrete representation (using pictures of ants and leaves), an abstract representation (using dots and patches) and a concreteness fading representation (fading from the ants to the dots). They found that students' transfer was better when they used a concreteness fading technique compared to concrete or abstract representations alone. Similarly McNeil & Fyfe (2012) found that undergraduates' transfer

of mathematical knowledge was also better when presented using a concreteness fading technique compared to abstract or concrete representations alone. Concreteness fading has also been shown to be effective with younger children. Fyfe, McNeil and Borjas (2015) found that 7 to 9 year-old children performed better on a transfer test after learning about mathematical equivalence in a concreteness fading condition (which involved presenting problems with physical objects followed by pictures of these objects and finally abstract numerical symbols) compared with learning from the physical objects alone, the abstract symbols alone or the concreteness fading condition in reverse.

One characteristic of learning from concrete representations, which has received comparatively little attention, is that many different concrete representations tend to be used together, particularly in the context of early number learning (e.g., a number book may display three dogs, four fish, and five pigs). It has been widely claimed that the use of multiple representations may lead to better learning outcomes in comparison to the use of a single representation (e.g., Ainsworth, 1999; Brenner et al., 1997; Jong et al., 1998; van der Meij & de Jong, 2006). Three advantages to the use of multiple representations have been proposed: two or more representations can provide complementary information; information in one representation can constrain the interpretation of information in another representation; and multiple representations can lead to the construction of deeper understanding by allowing learners to abstract information across different examples or representations (Ainsworth, 2006).

However, using multiple representations is not universally beneficial, and may come with costs (Ainsworth, 2006). Multiple representations can result in split attention and typically involve extraneous cognitive activities which may interfere with learning (Chandler & Sweller, 1992). Furthermore learners may fail to see how the representations are linked

to each other and fail to extract the key concept. For example, Ainsworth, Bibby, & Wood (2002) explored children's computational estimation using a computerized intervention that involved either pictorial representations, mathematical representations or both. Across two experiments they found that children learning with either pictorial or mathematical representations improved their estimation accuracy, but children working with multiple representations did not. The disadvantages of multiple representations are often interpreted within the framework of cognitive load theory: it is suggested that learners fail to benefit from multiple representations because they do not have sufficient cognitive resources (e.g. working memory capacity) to process the available information (Ainsworth, 2006; Ainsworth et al., 2002).

The multiple representations framework raises a question over whether the multiple representations typically involved in the concrete or concreteness fading approaches are beneficial. However, studies exploring the benefits or costs of multiple representations typically employ simultaneous presentation of multiple representations (e.g. pictures alongside text). In contrast, the multiple representations involved when learners are presented with multiple concrete examples or in the context of the concreteness fading approach are presented sequentially. We do not know whether the benefits of multiple representations outweigh the costs in terms of increased demands on cognitive resources when multiple representations are presented sequentially. Nevertheless, this is an important characteristic of the way that concrete representations are typically used that has received little consideration in the abstract vs. concrete debate to date.

Representations to support early number learning

These debates are pertinent to the question of what types of representations may

best support children's early number learning. As highlighted earlier, parents are frequently encouraged to support their children's early number learning using number books. Recently, Ward, Mazzocco, Bock, & Prokes (2016) investigated the types of number books that are targeted at parents with young children. They evaluated the structure and content of 120 such books, finding that 96% included at least one real-world set of items to be counted (e.g. apples), and 87% included only real-world pictures. Moreover, the nature of the items to be counted typically varied throughout the book (e.g. one horse, two sheep, three pigs etc.); in only 33% of the books reviewed did the identity of the items remain consistent. However, as we have seen, there is reason to question whether such representations would best support children's ability to learn numbers.

There is some direct evidence that using real-world pictures may introduce difficulties when children learn number words. Huang, Spelke, & Snedeker (2010) suggested that when children first learn the meaning of number words, they struggle to generalize from the real-world context in which it was taught. Huang et al. (2010) taught 16 three-year-old children about the number three. These children knew the meaning of the numbers one and two but had not yet mastered the meaning of the number three. They were trained using pictures of dogs and told, "This card has (does not have) three dogs". Children were then asked to select the card with three objects. They were successful when the test cards were pictures of dogs (in a different configuration or breed of dog to the training cards) but they could not successfully identify three when the pictures were of sheep. This study suggests that children may attach the meaning of number words to the context in which they were taught, particularly for smaller numbers, and consequently there may be a cost to using pictures of real-world items to teach children the meaning of number words.

Current models of early number learning propose that, as children learn the meaning of number words and symbols, they connect these with magnitude information (see Fazio, Bailey, Thompson, & Siegler, 2014 for review). Although debates surround the precise nature of the quantity information which underpins the meaning of number words and symbols (Leibovich, Katzin, Harel & Henik, 2016; Le Corre & Carey, 2007), the predominant model has focused on the role of the Approximate Magnitude System (AMS), a proposed cognitive system that allows individuals to represent the approximate quantity of a set of items (Feigenson, Dehaene & Spelke, 2004; Halberda, Mazzocco & Feigenson, 2008). Although many researchers prefer the term 'Approximate Number System' (ANS), as there is a debate about the extent to which this system is strictly numerical (e.g., Leibovich et al., 2017; Leibovich & Henik, 2013), we follow Lyons, Budgen, Zheng, De Jesus, and Ansari (2018) and use the more neutral term 'Approximate Magnitude System' (AMS). AMS representations are thought to be generated whenever an individual perceives a set of items. When children learn number words and symbols, it has been proposed that they become associated with these internal AMS representations. (To avoid confusion, we use the term 'AMS representation' to refer to an internal cognitive representation supported by the Approximate Magnitude System, and 'representation' to refer to an external representation of a mathematical idea, such as an array of dots or a number symbol. In other words a picture of five pizzas is a 'representation', and the cognitive representation generated when a child sees this picture is an 'AMS representation'.) Evidence commonly cited in support of this view comes from studies showing that the accuracy of both nonsymbolic and symbolic number comparisons are subject to ratio effects. That is to say that when asked to select the more numerous of two stimuli (either arrays of dots, or Arabic numerals) participants' accuracy decreases as the ratio between the stimuli approaches 1

(Dehaene, 2011; but see Lyons, Nuerk, & Ansari, 2015). Furthermore, children's performance on typical AMS acuity measures (i.e. dot comparison tasks) has been found to correlate with their school-level mathematics achievement (e.g., Halberda et al., 2008, but see Gilmore et al., 2013).

Huang et al.'s (2010) study of children's number learning focused on the learning of verbally presented number words, while much of the research exploring how numbers acquire their meaning has considered Arabic symbols. However, it has been suggested that number words and symbols are attached to meaning in the same way. Dehaene's (1992) triple code model incorporates separate codes for verbal and symbolic representations of number, but proposes that these are both mapped onto abstract magnitude representations (although the model leaves open the possibility that one of these mappings is dominant). Indeed, many theorists fail to draw a distinction, referring only to number symbols (words or digits) (e.g. Reynvoet & Sasanguie, 2016; Leibovich & Ansari, 2016; Piazza, 2010). The small number of studies to have explicitly distinguished words and digits have tended to find that children associate magnitude information with number words prior to Arabic digits (Bialystok, 1992; Knudsen, Fischer, Henning & Aschersleben, 2015; Von Aster & Shalev, 2007). When parents read numbers, children will typically receive both verbally-presented number words and visually-presented number symbols together, along with representations of the appropriate quantity.

The current study

Our goal was to explore whether the nature of the representations provided to a learner has an impact on their ability to learn symbolic number representations. In other words, might children attach AMS representations to number words and symbols more

effectively if they are matched with abstract representations, multiple concrete representations, or a combination of both? As discussed earlier, previous evidence about learning from representations provides mixed predictions about whether abstract or concrete representations will be more beneficial. Multiple concrete representations may support learning because they allow children to draw upon their real-world experiences of dealing with object sets and motivate children to learn. On the other hand, abstract representations may support learning because they do not include extraneous details and make it easier to identify and abstract the key information and therefore form the appropriate AMS representation. A concreteness fading approach incorporating both concrete and abstract representations may provide the benefits of both approaches. On the other hand use of either concreteness fading or multiple concrete representations may be less effective because the use of multiple representations leads to the overload of working memory.

Children begin to learn number words in their second year and are exposed to number words in combination with a wide range of different representations. Consequently it is challenging to address this question using Arabic numerals because everyone has different prior experiences with these symbolic representations. To study these processes in a more controlled fashion, researchers have therefore begun to use artificial symbol learning paradigms, in which participants learn novel symbolic representations after being exposed to these symbols in combination with nonsymbolic quantity representations, for example dot arrays (Lyons & Ansari, 2009; Lyons & Beilock, 2009; Merkley & Scerif, 2015; Merkley, Shimi, & Scerif, 2016; Zhao et al., 2012). Using this paradigm, it has been shown that adults can learn the meaning of novel symbols and subsequent performance on symbolic comparison tasks show characteristic ratio effects, suggesting that adults may be

learning these symbols in a similar way to how children are thought to learn Arabic number symbols (Merkley & Scerif, 2015). However, to our knowledge no research has investigated whether children can learn novel symbols by associating them with AMS representations, nor whether learning is affected by the nature of the examples with which novel symbols are paired in the training phase. In this paper we present three experiments that investigated the effects of abstract and multiple concrete representations on children's accuracy in learning novel number symbols.

Experiment 1

The aim of Experiment 1 was to investigate if children could successfully learn novel symbols using an artificial learning paradigm and to test whether children were more successful when presented with abstract or multiple concrete non-symbolic representations.

Method

Participants.

Seventy-four children ranging from 6 to 10 years old (M = 7.83 years, SD = 1.238, 35 boys) participated in the study. This study was powered to have 90% chance of detecting a small effect size (η_p^2 = .03, based on a correlation of 0.6 between performance in each condition). The children participated at 'Spring Scientist Week' at the University of Nottingham, an annual event run during the school holidays. Parents and children are invited to the university for half a day to take part in research activities and games. All studies were approved by the University of Nottingham School of Psychology Ethics Committee and all parents provided written consent for their child to participate. Children received a goody bag to thank them for taking part.

Procedure.

We used a within-subjects design where participants completed two training phases, in each of which they learned the meaning of five novel numerical symbols (10 in total). In one training condition, the novel symbols were paired with abstract representations while in the other training condition the novel symbols were paired with multiple concrete representations. The symbols used in each training condition were different. Each training phase was immediately followed by a symbolic comparison task. The order of the training conditions was counterbalanced across participants. Between the two training phases, participants took a pencil and paper arithmetic test. The training and comparison tasks were presented on a laptop computer using PsychoPy software (Peirce & Peirce, 2009) and the entire experiment took approximately 20 minutes.

Training.

In each of 100 passive training trials per condition, participants saw a symbol at the top of the screen with an array of dots/pictures (depending on whether they were in the abstract/multiple-concrete condition respectively) underneath, as shown in Figure 1. Children were asked to remember how many dots/pictures were associated with each symbol, but were not asked to respond.[1] To prevent counting, each trial appeared for only 1000ms with a blank screen for 200ms between each trial. The trials were presented in random order and participants received breaks every 25 trials.

Ten symbols were selected from the LaTeX amsmath package, so that they would be unfamiliar to children of this age. Five symbols were used for the multiple-concrete condition and a different five symbols used for the abstract condition. Each symbol was

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[1] The exact wording used was "Hello. My name is Ali the Alien. I am going to show you some symbols from my planet. These symbols mean different amounts. I am going to show you the symbol at the top of the page and the amount that each symbol means underneath in dots. Can you try and remember the amount these symbols represent in my world? Are you ready?"

associated with either 5, 10, 15, 20, or 25. Children saw each symbol 20 times, each time paired with a numerically equivalent but spatially different array.

In the abstract condition the nonsymbolic stimuli were arrays of dots. The dots, which were the same size in all trials, were randomly placed within a 10×10 grid to create 20 different displays per numerosity. Each display was only presented once in the experiment. In the multiple-concrete condition, the stimuli were arrays of identical pictures. The pictures were the same size as the dots and were placed in exactly the same places as the dots in the abstract condition. A total of 20 different pictures that would be familiar to children of this age were selected (e.g. frogs, pizza, cars). In line with the dominant approach adopted by children's number books, each picture was only used once for each number. For example, the novel symbol for 5 was displayed once each with a set of 5 frogs, 5 pizzas, 5 cars etc.

Symbolic magnitude comparison task.

To assess how accurately participants associated magnitudes with our novel symbols we used a symbolic comparison task. Such tasks have commonly been used to assess numerical processing skills, including the precision of AMS representations (e.g., De Smedt, Noël, Gilmore, & Ansari, 2013; Holloway & Ansari, 2009). Performance on Arabic symbolic comparison tasks typically predicts school-level mathematics achievement (Schneider et al., 2017; Vanbinst, Ansari, Ghesquière, & De Smedt, 2016).

Immediately following each training phase participants completed a symbolic comparison task to assess whether they had learnt the numerical meaning of the novel symbols. On each of 40 trials they were presented with two symbols, and asked to select the one that represented the larger number. The symbols were presented until the participant responded by pressing a key on the keyboard. Every combination of the 5 symbols in each condition was presented four times with display side counterbalanced.

While some researchers have used numerical ratio effects (NREs) or Weber fractions to index performance on numerical comparison tasks, we have found that accuracy measures typically show superior psychometric properties (Inglis & Gilmore, 2014). Therefore, we calculated a mean accuracy score for the abstract and concrete conditions as well as an overall mean accuracy.

Arithmetic test.

Between the training conditions participants completed the Woodcock Johnson arithmetic fluency test, which requires participants to answer as many one- and two-digit sums as possible in 3 minutes. Raw scores (total correct) were used in the analysis.

Results and Discussion

We first assessed whether children were able to learn the meaning of the novel symbols and whether performance on the symbolic magnitude task showed characteristics typically observed on Arabic symbolic comparison tasks. We then compared performance for the two different training conditions.

Children were significantly more accurate than chance (0.5) in both the abstract, $M=$ .72, SD = .166, $t(73)$ = 11.41, $p < .001$, and multiple-concrete conditions, M = .66, SD = .195, $t(73)$ = 7.25, $p < .001$, indicating that they were, to some extent at least, able to engage with learning the meaning of the novel numerical stimuli.

To test whether children had learned the meanings of the full range of symbols or had simply learnt the symbols for the smallest and largest quantities, we examined children's performance on the symbolic magnitude comparison test after removing trials including the smallest and largest symbols. The accuracy for both the abstract (M = .57, SD = .221) and multiple-concrete (M = .61, SD = .227) conditions were still significantly above

chance, $t(73) = 2.63$, $p = .010$, $t(73) = 4.02$, $p < .001$, respectively, indicating that children

had not just learned the meaning of the two extreme values.

The ratio between the two comparison symbols in this task ranged from .2 to .8. If

children had mapped the novel symbols onto AMS representations of quantity then we

would expect a significant effect of ratio on performance. We evaluated this by conducting a

by-items linear regression, predicting the proportion of participants correctly responding to

each trial by the trial's ratio (calculated as smaller:larger). This revealed a significant effect

of ratio, $\beta = -.744$, $p < .001$, $R^2 = .554$. Overall accuracy was correlated with performance on

the Woodcock Johnson arithmetic fluency test, $r = .391$, $p = .001$, which remained significant

after controlling for age, $pr = .241$, $p = .040$. Therefore, performance on the symbolic

comparison task showed performance characteristics that are typically observed on both

Arabic symbolic comparison tasks and nonsymbolic comparison tasks.

To explore the differing effects of abstract and multiple concrete representations on

symbol learning, we conducted a 2×2 Analysis of Covariance (ANCOVA) with condition

(abstract or multiple-concrete) as a within-subjects factor, order (abstract condition first or

multiple-concrete condition first) as a between-subjects factor and, because participants

spanned a large age range, age as a covariate. This revealed a significant condition by order

interaction effect, $F(1, 71) = 7.43$, $p = .008$, $\eta_p^2 = .095$, as shown in Figure 2. As expected,

there was also a main effect of age, $F(1,71) = 10.736$, $p = .002$, $\eta_p^2 = .131$. Neither the main

effects of condition and order, nor the condition by age interaction effect were significant,

Fs < 1. For the group who completed the abstract condition first there was no significant

difference between accuracy in the abstract and multiple-concrete conditions, $t(37) = .235$,

$p = .815$. However for the group who completed the concrete representations first, scores in

the abstract condition were significantly higher than those in the multiple-concrete

condition, $t(35) = 3.361$, $p = .002$. Finally, in a between-subjects comparison, we compared scores for only the condition each participant completed first. This revealed that those who learned from abstract representations scored higher than those who learned from multiple concrete representations (Ms = .70, .62; SDs = .16, .20 for the abstract and multiple-concrete conditions respectively), but this difference did not reach significance, $t(71) = 1.878$, $p = .065$; although we note that this study was powered for a within-subjects comparison not a between-subjects comparison.

In sum, we found three main results. First, children did seem able to learn the meaning of novel number symbols following a short training session: the subsequent symbolic comparison task showed above-chance performance. Second, children's performance when comparing the newly learned symbols showed similar ratio effects to those found in typical Arabic symbolic comparison tasks. This pattern of results is consistent with the suggestion that the children mapped the novel symbols to AMS representations in a similar fashion to that proposed by the dominant account of Arabic symbol acquisition (e.g., Fazio et al., 2014). Note that while some have argued that the ratio effects observed in Arabic symbolic comparison tasks might be caused by non-magnitude factors such as relative word frequency or unit notation (e.g., Lyons et al., 2015), this does not seem likely here. Since we used an artificial symbol paradigm, the only way in which our participants could have performed at above-chance levels would be by forming an association between the novel symbols and the magnitudes of the dot arrays presented during the learning phase.

Finally, we found that children seemed to be more successful when they learned symbols using abstract representations than when they learned using multiple concrete representations, at least in the case of children who learned from the concrete condition

first. This finding is perhaps surprising in view of the suggestion that concreteness fading – using multiple concrete representations first followed by abstract representations – is an effective method. However, children in this condition did not experience true concreteness fading, where concrete representations are gradually withdrawn in favor of abstract representations. Instead they learned two different symbol systems. To overcome this limitation, we conducted a further experiment.

Experiment 2

The main aim of Experiment 2 was to investigate whether concreteness fading would lead to more effective learning of new number symbols, compared to learning with abstract representations or multiple concrete representations. We also included an abstractness fading condition (concreteness fading in reverse, sometimes referred to as 'concreteness introduction') in a between-subjects experiment.

Method

Before data collection commenced, the study hypotheses, design, sample size, exclusion criteria and analysis plan was pre-registered at AsPredicted.com. The pre-registered protocol is available at https://aspredicted.org/8wp6w.pdf

Participants.

Participants were a new group of 216 children (mean age 8 years, range: 7 years 1 month to 9 years 1 month, 114 boys). This sample size was chosen to give 90% power to detect a medium effect of η_p^2 = 0.06. Children were recruited from three primary schools in Nottinghamshire, UK, which had varying socio-economic status (SES), based on free school meals eligibility: 157 children came from two low-SES schools and 59 children came from a high-SES school. Children received stickers to thank them for taking part. Ethical approval

for the study was received from the Loughborough University Ethics Approvals (Human Participants) Sub-Committee.

Procedure.

We used a between-subjects design where participants were randomly assigned to one of four conditions: abstract, multiple-concrete, concreteness fading and abstractness fading. In each condition participants completed a training session, followed by a symbolic comparison task. The training and comparison tasks were presented on a laptop computer using PsychoPy software (Peirce & Peirce, 2009).

Training.

The training phase consisted of 200 trials in which participants saw a symbol at the top of the screen with an array of dots/pictures (depending on condition) underneath, as shown in Figure 1. Children were asked to learn how many dots/pictures were associated with each symbol. To prevent counting, each trial was displayed for 1000ms with a blank screen for 200ms between each trial. The trials were presented in random order and participants received breaks every 20 trials.

Five of the same novel numerical symbol stimuli from Experiment 1 were used. Again, these symbols represented the numerosities 5, 10, 15, 20 and 25. To ensure that there was no inherent magnitude information included in the symbols, the association between the symbols and numerosities was counterbalanced across participants.

The dot/picture arrays were the same as those used in Experiment 1. Children in all conditions saw 200 training trials. In the abstract and multiple-concrete conditions children saw the same 100 trials as Experiment 1 but each trial was presented twice, in a random order. In the concreteness fading and abstractness fading conditions participants saw 100

dot arrays and 100 picture arrays. The combination of dot and picture arrays used in each condition is shown in Table 1.

Symbolic magnitude comparison task.

Immediately following the training phase participants completed a symbolic magnitude comparison task, which was identical across conditions and identical to that used in Experiment 1, other than each stimuli pair was presented 8 times giving a total of 80 trials. Participants were given a short break half-way through the trials. Mean accuracy was calculated for each participant.

Results and Discussion

All children completed the full experiment and no-one met our pre-registered exclusion criteria of performance more than 3 SDs above or below the mean. Therefore all analyses were performed on the full pre-registered sample of 216.

Mean accuracy was significantly above chance for all conditions, abstract $M = .69$, $SD = .19$, $t(53) = 7.34$, $p < .001$; multiple-concrete, $M = .58$, $SD = .14$, $t(53) = 4.34$, $p < .001$; abstractness fading, $M = .56$, $SD = .16$, $t(53) = 2.69$, $p = .009$; concreteness fading, $M = .56$, $SD = .14$, $t(53) = 3.12$, $p = .003$. This replicates and extends Experiment 1 by demonstrating that children can learn the meaning of novel numerical symbols from a training session with either abstract, concrete or a mixture of both abstract and concrete representations. Next we checked whether children's performance on the symbolic comparison task showed the canonical ratio effect by conducting a by-items linear regression predicting the proportion of children answering each problem correctly by the problem's ratio. This revealed a significant effect of ratio, $β = -.839$, $p < .001$, $R^2 = .703$.

A one-way between-subjects Analysis of Variance (ANOVA) was conducted to compare mean accuracies from the four training conditions. This revealed a significant

effect of training condition on accuracy, $F(3, 212) = 8.05$, $p < .001$, $\eta_p^2 = .102$, shown in

Figure 3. Post hoc comparisons using Tukey HSD tests indicated that the mean score for the

abstract condition was significantly higher than the mean score for the multiple-concrete

condition ($p = .004$), the abstractness fading condition ($p < .001$) and the concreteness

fading condition ($p < .001$). There were no significant differences between any other

conditions (all descriptive statistics are given above).[2,3]

In sum, as in Experiment 1, we found that children learned novel symbols more

effectively when they were presented with abstract representations alone than when they

were presented with multiple concrete representations. However, here we also found that

abstract representations alone were more effective than a mixture of abstract and concrete

representations, using both the concreteness fading and the abstractness fading techniques.

Performance did not differ between the concreteness fading and abstractness fading

conditions.

In both Experiments 1 and 2 we found that abstract representations appeared to

help children acquire the meaning of novel number symbols. But why might this be the

case? In both these studies the presence of concrete representations was confounded with

the presence of multiple representations: as in traditional children's number books, we

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[2] In a non-preregistered analysis suggested by a reviewer, we also conducted an ANCOVA that included age as a covariate. As in the main analysis, this revealed a significant effect of training condition on accuracy, $F(3, 211) = 8.25$, $p < .001$, $\eta_p^2 = .105$. There was no significant effect of age, $F < 1$.

[3] Although we pre-registered a sample size of 216, we in fact tested 259 due to the number of children in the schools who wanted to take part. Running the analysis on this larger sample of $N = 259$ yielded essentially identical results. There was a significant effect of training condition on accuracy, $F(3, 255) = 9.42$, $p < .001$, $\eta_p^2 = .100$. Post hoc comparisons indicated that the mean score for the abstract condition ($M = .69$, $SD = .19$) was significantly higher than the mean score for the multiple-concrete condition ($M = .58$, $SD = .14$, $p = .002$), the abstractness fading condition ($M = .55$, $SD = .15$, $p < .001$) and the concreteness fading condition ($M = .56$, $SD = .15$, $p < .001$). There were no significant differences between any other conditions.

presented multiple concrete representations in the concrete conditions. In other words, children learned to associate the new symbol for 5 with five frogs, five pizzas, five cars and so on. In contrast, in the abstract condition children learned to associate this symbol with many arrays of five dots. This observation leaves open two possible explanations for the abstract advantage we found across Experiments 1 and 2: first, this could be due to the use of abstract rather than concrete representations, or alternatively this could be the result of using a single representation for numerosities – dots – rather than multiple representations. To disentangle the effect of concrete representations from the effect of multiple representations, we ran a third experiment.

Experiment 3

The goal of Experiment 3 was to compare the effectiveness of learning novel number symbols from a single concrete representation compared to multiple concrete representations. In a between-subjects experiment we compared three learning conditions, where children learned new symbols with abstract representations, single concrete representations (five fish, five fish, ten fish, ten fish, etc.) and multiple concrete representations (five fish, five cakes, ten rockets, ten cars, etc.). If concrete representations per se disadvantage children, then we would expect to see the same abstract advantage found in Experiments 1 and 2. If, however, the abstract advantage was the result of the multiple-concrete condition using multiple representations, we would expect to see children in the abstract and single-concrete conditions outperform those in the multiple-concrete condition.

Method

The study hypotheses, design, sample size, exclusion criteria and analysis plan was pre-registered at AsPredicted.com. The pre-registered protocol is available at https://aspredicted.org/tx92d.pdf

Participants.

A new sample of 120 children took part in this study. This sample size gave 90% power to detect an effect of η_p^2 = .1 (based on the effect size found in Experiment 2). The children were recruited from two primary schools in Nottinghamshire, UK, which were both of low socio-economic status (SES), based on free school meal eligibility. Children's ages ranged from 7 years 4 months to 9 years 3 months (M = 8 years 3 months, 52 boys). Children received stickers to thank them for taking part. Ethical approval for the study was received from the Loughborough University Ethics Approvals (Human Participants) Sub-Committee.

Procedure.

Participants were randomly assigned to one of three conditions: abstract, single-concrete and multiple-concrete. Participants in the single-concrete condition were then randomly assigned to one of four sets of stimuli (fish, cake, rocket or cars). In each condition participants completed a training session followed by a symbolic comparison task. The training and comparison tasks were presented on a laptop computer using PsychoPy software (Peirce & Peirce, 2009).

Training.

Participants completed a similar training phase to that in Experiment 2. In each of 200 trials participants saw a symbol at the top of the screen with an array of dots/pictures (depending on condition) underneath. Children were asked to learn how many dots/pictures were associated with each symbol. To prevent counting, each trial was displayed for 1000ms

with a blank screen displayed for 200ms between each trial. The trials were presented in random order and participants received breaks every 20 trials.

The same five novel numerical symbol stimuli from Experiment 2 were used. Again, these symbols represented the numerosities 5, 10, 15, 20 and 25. To ensure that there was no inherent magnitude information in the symbols, the association between the symbols and numerosities was counterbalanced across participants.

New nonsymbolic arrays were created. In the abstract condition the nonsymbolic stimuli were arrays of dots. The dots, which were the same size in all trials, were randomly placed within a 10×10 grid to create 20 different displays per numerosity. The multiple-concrete condition was the same as in Experiments 1 & 2. The concrete stimuli were arrays of pictures, which were the same within an array but varied across the arrays. Twenty arrays of different pictures were created for each of the 5 numerosities by placing pictures of the same size as the dots in the exactly the same position as the dots stimuli. In the single-concrete condition the pictures were the same both within and across arrays. Four different sets of the stimuli were created for the single-concrete condition and participants were randomly assigned to one. Each set of stimuli used a different picture (fish, cake, rocket or car) and contained 20 arrays of pictures for each of the five numerosities. These were created by placing pictures of the same size as the dots in the exactly the same position as the dots stimuli. In all conditions each trial was presented twice, in random order, resulting in 200 training trials.

Symbolic magnitude comparison task.

Children were presented with the same symbolic magnitude comparison task used in Experiment 2.

Results and Discussion

All children completed the full experiment and no-one met the pre-registered

exclusion criteria of performance more than 3 SDs above or below the mean. Therefore all

analyses were performed on the full pre-registered sample of 120.

Participants performed at above-chance levels in all three conditions (abstract, M =

.70, SD = .16, $t(39)$ = 7.92, $p < .001$; single-concrete M = .67, SD = .16, $t(39)$ = 6.39, $p < .001$;

multiple-concrete, M = .56, SD = .14, $t(39)$ = 2.96, p = .005). Thus we again replicated the

finding that children can accurately learn the meaning of novel symbols by associating them

with the magnitude of nonsymbolic abstract or concrete representations. As before we

conducted a by-items linear regression to assess whether there was a canonical ratio effect

and, again as before, we found a significant effect of ratio, β = -.545, $p < .001$, R^2 = .296.

A one-way between-subjects ANOVA was conducted to compare accuracy on the

symbolic comparison task following the three training conditions. There was a significant

effect of condition on accuracy, $F(2, 117)$ = 8.66, $p < .001$, η_p^2 = .129, shown in Figure 4. Post

hoc Tukey HSD tests indicated that the mean score for the multiple-concrete group was

significantly lower than for the abstract ($p < .001$) or single-concrete groups (p = .010). These

latter two groups did not differ significantly (p = .566; descriptive statistics are given

above).[4]

In sum, we again found that children who learned the novel symbols using abstract

representations outperformed those who learned using multiple concrete representations.

However, we found that there was no significant benefit to learning from abstract

representations compared to single concrete representations, or vice versa. This pattern of

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[4] In a non-preregistered analysis suggested by a reviewer, we also conducted an ANCOVA
that included age as a covariate. As in the main analysis, this revealed a significant effect of
condition on accuracy, $F(2, 112)$ = 8.66, p = .001, η_p^2 = .121. There was no significant effect
of age, $F < 1$.

results suggests that children's difficulty with the concrete stimuli used in Experiments 1 and 2 stemmed not from their concreteness per se, but rather from the difficulty of dealing with multiple representations across trials.

General Discussion

Summary of Main Findings

Prior research has found that there are individual differences in preschoolers' number skills, and that these differences are predictive of later mathematical achievement (e.g., Ginsburg et al., 2008; Melhuish et al., 2008). It is therefore important that research investigates the most effective ways of supporting young children to develop number skills. In particular, how can we help children to attach magnitudes to number words and symbols? Here we focused on the relative merits of learning novel number symbols using abstract and multiple concrete representations.

In Experiment 1 we found an advantage for single abstract representations: children who learned the meaning of novel symbols by pairing them with numerosities represented by arrays of dots performed better on a subsequent symbolic comparison task than those who paired them with multiple concrete representations. Experiment 2 replicated this result and extended it by also demonstrating an advantage for single abstract representations over both concreteness fading and abstractness fading approaches, each of which involved a mixture of abstract and multiple concrete representations. Finally, Experiment 3 demonstrated that the advantage for abstract representations found in Experiments 1 and 2 was not due to abstract representations being inherently superior to concrete representations, but rather was due to the use of multiple concrete representations in the concrete conditions. In Experiment 3 we found no significant difference in symbolic comparison performance between those who learned from dots and those who learned

from a single concrete representation. But both these groups outperformed those who learned with multiple concrete representations.

Overall, we found that there is a cost to multiple representations. Learning number symbols from multiple concrete representations – the approach adopted by the majority of children's number books (Ward et al., 2016) – seems to be less effective than learning from a consistent concrete representation or a consistent abstract representation. Our discussion of these findings falls into three main sections. First, we discuss possible cognitive mechanisms for these results, focusing on the 'seductive details' effect. Second, we draw out implications for early number learning, and particularly discuss how our experimental setting differed from that in which children typically encounter Arabic numerals for the first time. Finally, we discuss implications for the wider debate about whether abstract or concrete representations should be favored when teaching mathematics.

Mechanisms

Why did those children who learned from single representations outperform those who learned from multiple representations? The so-called 'seductive details' effect provides a natural account. Prior research has found that seductive details – the provision of information unconnected to the learning goal – can harm learning by activating irrelevant prior knowledge that the learner may try to integrate with the to-be-learned knowledge (e.g., Harp & Mayer, 1998). For instance, showing a child an array of five frogs may bring to mind knowledge about frogs that is irrelevant to the fiveness of the representation. If the child is to be successful, then this irrelevant prior knowledge must be inhibited. The seductive details account also seems to explain the difference in performance between the multiple- and single-concrete conditions seen in Experiment 3. It is likely that inhibiting prior

knowledge is easier when the same knowledge is activated on every trial than when new knowledge is activated from trial to trial. For example, performance on trial n of a Stroop task is facilitated when the to-be-inhibited text is identical to that presented on trial n - 1 (Lowe, 1979; MacLeod, 1991).

Many researchers have found that the failure to inhibit irrelevant prior knowledge can damage learning by consuming limited working memory capacity (e.g., Harnishfeger & Bjorklund, 1993; Sanchez & Wiley, 2006). These factors therefore suggest that one plausible account of the lower performance of children in the multiple-concrete conditions was that they failed to inhibit prior knowledge automatically activated by the concrete representations, that this increased the load on their working memory, and that this therefore damaged their ability to map the novel symbols onto their AMS representations.

Another possibility is that children's AMS representations themselves were less precise in the multiple-concrete conditions, due to an increase in working memory load caused by a failure to successfully inhibit irrelevant prior knowledge activated by the concrete representations. The literature offers conflicting evidence about the plausibility of this latter account. Some researchers have found that performance on nonsymbolic comparison tasks is correlated with measures of working memory capacity, suggesting that working memory resources are implicated in the ability to form precise AMS representations (Xenidou-Dervou, De Smedt, van der Schoot, & van Lieshout, 2013; Xenidou-Dervou, Molenaar, Ansari, van der Schoot, & van Lieshout, 2017). This conclusion is also consistent with the finding that working memory loads damage participants' ability to perform nonsymbolic arithmetic using AMS representations (Xenidou-Dervou, van Lieshout, & van der Schoot, 2014). On the other hand, Fuhs, McNeil, Kelley, O'Rear, & Villano (2016) asked children to tackle a nonsymbolic comparison task using stimuli similar to those used

in our abstract and multiple-concrete conditions. Children were asked to select the more numerous of two arrays of either dots or pictures. Unlike the present study children were not asked to associate the arrays with numerical symbols. Fuhs and colleagues found that nonsymbolic comparison performance did not significantly differ between the abstract and multiple-concrete conditions, suggesting that children are able to form similarly precise AMS representations from both stimuli types. This would be surprising if the lower performance in our study were primarily due to imprecise AMS representations. A third possibility is that both these accounts – less precise AMS representations and lower quality AMS-to-symbol mappings – played a role in the lower performance exhibited by the children in the multiple-concrete conditions.

Here we compared learning from single abstract representations with learning from multiple or single concrete representations. We did not consider learning from multiple abstract representations (e.g. arrays of different colored dots). If the seductive details account is the mechanism behind our findings then we would expect learning from multiple abstract representations to be more effective than learning from multiple concrete representations. Unlike multiple concrete representations, multiple abstract representations would not activate learners' prior experiences and there would be less extraneous information to be inhibited. However, this remains to be tested.

Implications for Early Number Learning

Parents are commonly encouraged to help their children acquire number words and symbols by reading number books. What implications do our findings have for the design of such resources? We highlight two important differences between the context of the experiments reported here and children's first introduction to Arabic number symbols.

First, typically children first encounter Arabic number symbols at a much younger age than the participants in our study (who ranged from 6 to 10 years old). Clearly some care is needed before we can generalize the lessons learned from how older children performed on our artificial symbol learning paradigm to the learning of Arabic symbols by younger children. However, if the mechanism behind our results is as we have suggested, then there are two reasons to suppose that the abstract advantage would be even greater with younger children. Earlier research has found that the harmful effects of seductive details are greater for participants with lower working memory capacity (Sanchez & Wiley, 2006). Since working memory capacity is developmental (e.g., Gathercole, Pickering, Ambridge, & Wearing, 2004), we could reasonably suppose that younger children would be more distracted by irrelevant knowledge when reading number books than older children, and therefore that multiple concrete representations would have a more deleterious effect on their number symbol acquisition.

Second, the older participants in our study were all familiar with the notion of representing numerosities with symbols, a fact which allowed us to simply tell them that the novel symbols were related to the number of items in the display. In contrast, younger children encountering Arabic symbols or number words for the first time must first infer that the concept the symbol represents is the number of items in the display, and not some property of the objects represented in the concrete representations. Indeed, as discussed above, Huang et al. (2010) found that younger children sometimes find it difficult to generalize number words from the real-world contexts in which they were introduced.

Both these factors suggest that the abstract single-representation advantage we found here may be even greater in the context of young children learning to associate Arabic symbols or number words to AMS representations for the first time. Indeed, it is

notable that the nature of the nonsymbolic representations had an effect on number symbol learning even when participants were explicitly told to focus on quantity. However, the fact that the children in these experiments already knew number words for the quantities represented means that these artificial learning experiments are not a perfect model of early number learning. Directly investigating how the nature of representations affect early number learning in young children would be a valuable goal for future research.

These considerations highlight a further issue. In all three experiments we used children's symbolic comparison performance as a measure of the extent to which they had learned the novel symbol system. However, it is also important that children are able to map between number symbols and nonsymbolic quantities. In other words, although we would certainly like children to understand that 5 is greater than 3 – the skill we tested – we would also like them to know that the symbol '5' and number word 'five' should be attached to a picture of five cars rather than a picture of three cars. Would the advantage for single representations we found here generalize to alternative measures of numerical performance such as mapping tasks? We cannot directly answer this question, but do note that there is evidence that Arabic symbolic comparison seems to be a more important skill for formal mathematics than symbolic-to-nonsymbolic mapping. Mundy and Gilmore (2009) found that 6-7 year old children's symbolic comparison performance was strongly correlated with a test of school mathematics achievement ($r = .53$), whereas performance on a mapping task was not significantly correlated with the same test ($r = .17$). In other words, associating AMS representations with number symbols in such a way that permits fluent symbolic comparison appears to be more important for children's future mathematical development than mapping. We found that performance was impaired when using multiple concrete representations.

Abstract and Concrete Representations in Mathematics Learning

There is a longstanding debate about whether instructional materials in mathematics should favor abstract or concrete representations. While many teachers and researchers have argued in favor of using concrete representations (e.g., Bruner, 1966; Piaget, 1971), others have pointed out that there are reasons to prefer abstract representations (e.g., Kaminski et al., 2008). Still others have proposed combining both abstract and concrete representations using a concreteness fading technique (e.g., Fyfe et al., 2014).

Our results are clear. In the context of associating numerosities with novel symbols, we found that children who learned from abstract representations outperformed those who learned from either multiple concrete representations, or from a sequence of multiple representations that faded from concrete to abstract (or vice versa). These results, combined with those from the wider literature, perhaps suggest that looking for a universal answer to the abstract versus concrete debate may be misguided. For instance, Day et al. (2015) found that abstract representations were more effective than concrete representations when teaching beginning psychology undergraduates about measures of central tendency. Koedinger and Nathan (2004) found that high school student's algebra problem solving performance was improved when using concrete story problems opposed to abstract mathematical equations. McNeil and Fyfe (2012) found that concreteness fading improved undergraduates' learning of modulo arithmetic compared to the use of concrete and abstract representations alone. One way of making sense of these disparate findings is to propose that there is no universal answer to the question of what type of representations are better for learning in general. It may be that different answers will emerge for young children learning number symbols to high school students solving algebra problems, to

undergraduates learning mathematical concepts. If this suggestion is correct, then it would be beneficial for researchers to move beyond the question of *whether* abstract or concrete representations are better, and instead to ask *when* they are better.

Consistent with this suggestion, the result of Experiment 3 demonstrated that where concrete representations have been found to be less effective than abstract representations, this may not be down to the concrete nature of the representations per se, but rather to the use of multiple concrete representations rather than a single abstract representation. This finding echoes Ainsworth's (2006) warning that although multiple representations can sometimes be beneficial for learning, this is not always the case. Apparently sometimes it might be best to keep things simple.

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	Blocks of 20 trials				
	1 & 2	3 & 4	5 & 6	7 & 8	9 & 10
Abstract	100% Abstract	100% Abstract	100% Abstract	100% Abstract	100% Abstract
Multiple concrete	100% Multiple concrete	100% Multiple concrete	100% Multiple concrete	100% Multiple concrete	100% Multiple concrete
Concreteness fading	100% Multiple concrete	75% Multiple concrete	50% Multiple concrete	25% Multiple concrete	0% Multiple concrete
	0% Abstract	25% Abstract	50% Abstract	75% Abstract	100% Abstract
Abstractness fading	0% Multiple concrete	25% Multiple concrete	50% Multiple concrete	75% Multiple concrete	100% Multiple concrete
	100% Abstract	75% Abstract	50% Abstract	25% Abstract	0% Abstract

Table 1: Combinations of abstract and multiple concrete arrays used for each block in all four conditions in Experiment 2.
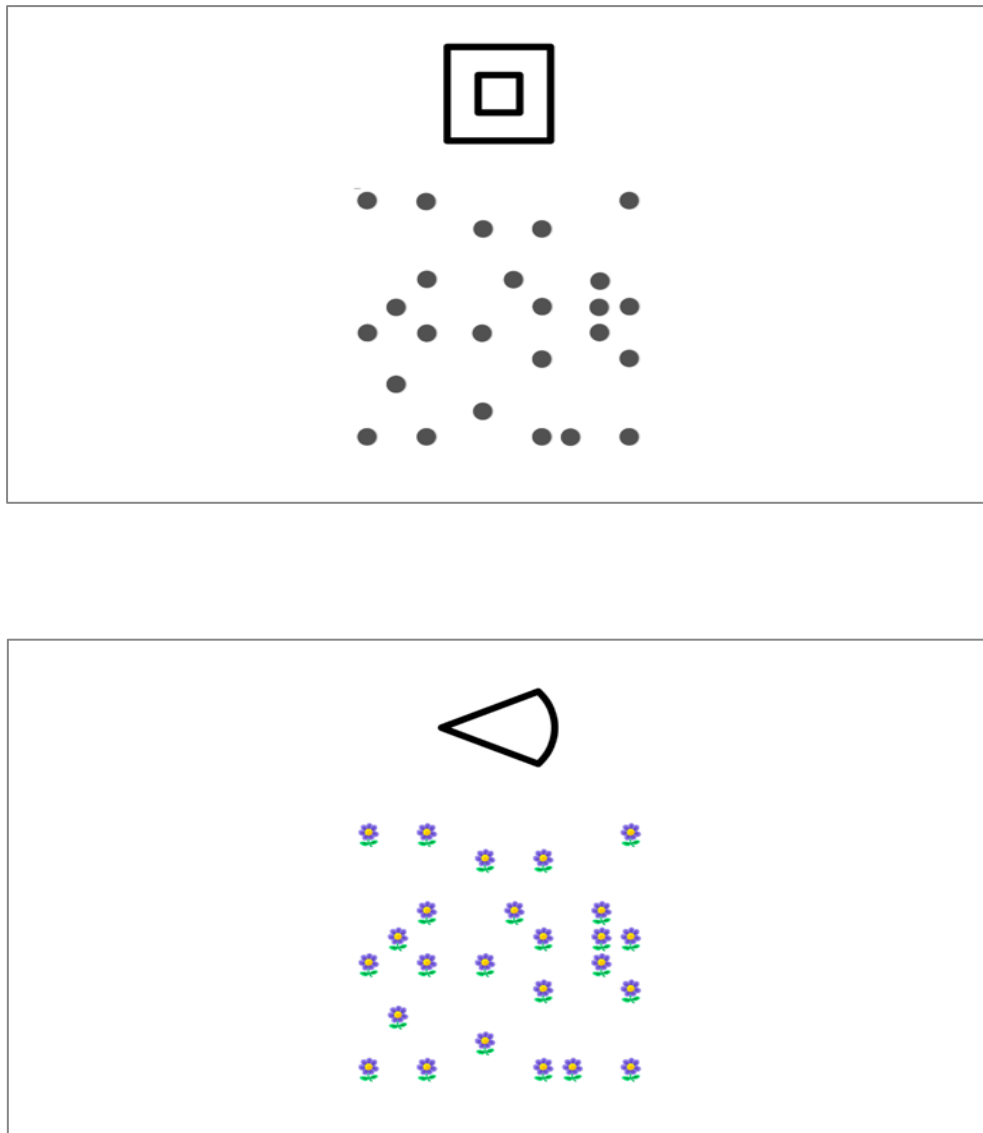
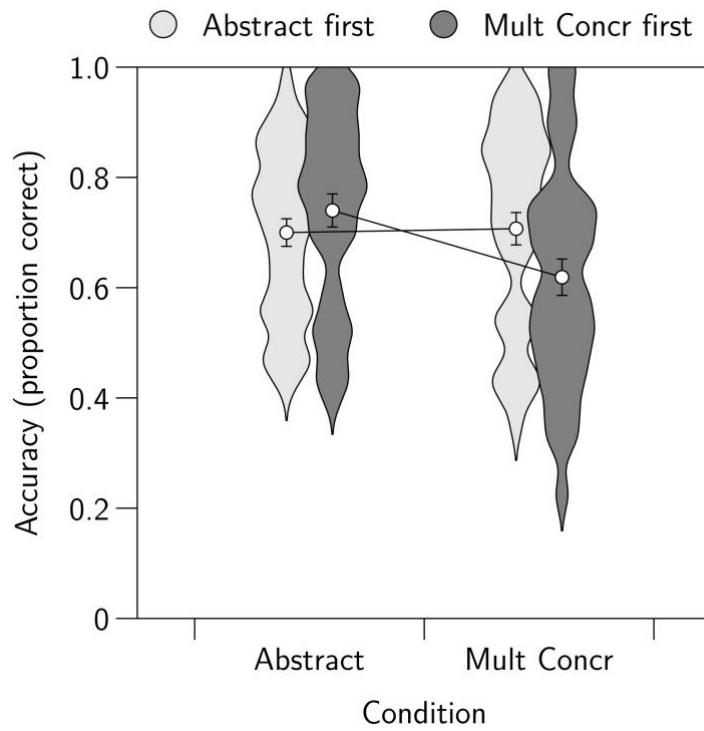Figure 1. Examples of a) abstract and b) concrete stimuli used in the training phase.

Figure 2. Violin plots showing accuracies on the symbolic comparison task in Experiment 1, by condition and order. Points show the mean in each condition, error bars show ± 1 SE of the mean.
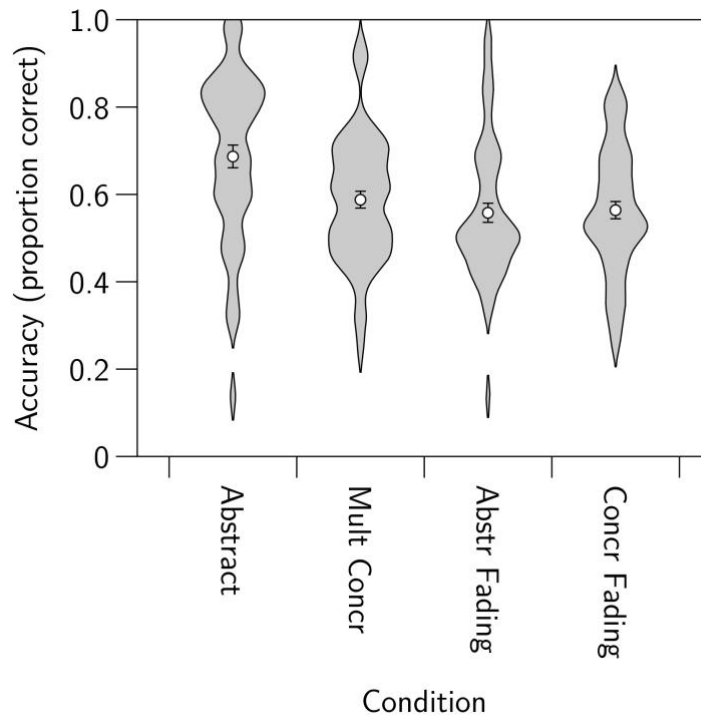
Figure 3. Violin plots showing accuracies on the symbolic comparison task in Experiment 2, by condition. Points show the mean in each condition, error bars show ± 1 SE of the mean.
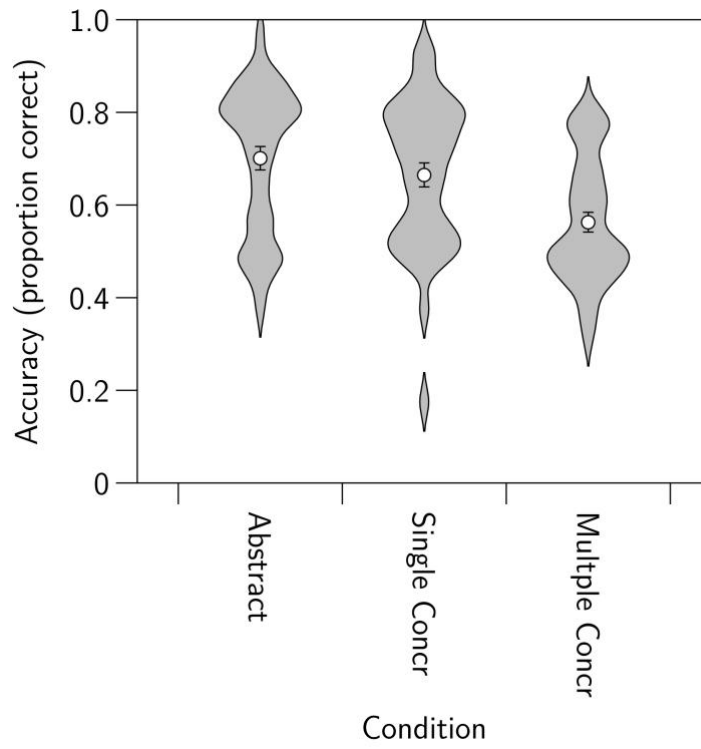
Figure 4. Violin plots showing accuracies on the symbolic comparison task in Experiment 3, by condition. Points show the mean in each condition, error bars show ±1 SE of the mean.