

Self-Explanation Training Improves Proof Comprehension: Supplementary Materials

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This document contains supplementary materials associated with the paper:

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Other materials referred to in the paper (self-explanation training, Proofs A, B and C, and open-ended questions for Proof A) can be found in the paper's Appendix.

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1 Scoring Scheme for Comprehension Test A (Experiment 1)

The list below shows the open-ended questions that constituted the comprehension test for Proof A in Experiment 1. For each question, the list gives a brief answer together with notes to the graders on things not to accept (note that in British English it is common to use 'mark' to mean 'point'). Question order was randomised for each participant.

1. What does it mean for a number to be triadic?

A number is triadic if it can be represented in the form $4k + 3$ (one mark) for some integer k (one mark).

Do not accept: A number is triadic if it can be represented in the form $4j + 1$ for some integer j .

2. What does it mean for a number to be a prime?

A number is prime if it is greater than 1 (one mark) and is divisible only by 1 and itself (one mark).

An answer of the form: 'A number is prime if it is divisible by 1 and itself' will score one mark only.

Do not accept: A number is prime if it is 1, 2, 3, 5, 7, 11.

3. What kind of proof is this (e.g. Proof by induction)?

This is a proof by contradiction (one mark).

Do not accept: This is a proof by induction, This is a direct proof etc.

4. In line 3, what is the purpose of assuming that the theorem is false and that there are only infinitely many triadic primes?

The purpose is to set up a proof by contradiction (one mark). We assume that the theorem is false (one mark) and prove this cannot possibly be the case thus showing the theorem has to be true (one mark).

Do not accept: The purpose is so we can use monadic primes rather than triadic primes as they are easier to work with in a proof.

5. In line 5, why does the fact that 3 does not divide $4p_2 \dots p_n$ imply that 3 does not divide M ?

Because, by rules of division, we know that if 3 does not divide $4p_2 \dots p_n$, it cannot divide M since $M = 4p_2 \dots p_n + 3$ (2 marks).

6. Which claim(s) in the proof logically depend on line 2 of the proof, the claim that the product of monadic numbers is monadic?

The claim that M itself is monadic because M is made up of a product of monadic numbers (2 marks). Also accept Line 8.

Do not accept: The claim that p_2, \dots, p_n do not divide M and hence 3 does not divide M .

7. Which of the following summaries best captures the ideas of the proof:

(a) It assumes there are infinitely many triadic primes and uses them to construct a triadic number M that has only monadic prime factors, which would imply M is also monadic. This cannot be true as M is triadic and thus the theorem is proved.

(b) It lets $M = 4p_2 \dots p_n + 3$, where p_i are prime numbers and $p_1 = 3$. Thus, 2 does not divide M because M is odd. Further, p_i does not divide M because it leaves a remainder of 3.

(c) The proof introduces monadic primes to be used later on in the proof. It lets $M = 4p_2 \dots p_n + 3$ and shows 2 does not divide M , since 2 is even and M is odd. It then uses monadic primes to create an infinite triadic prime.

(a) is the correct selection (1 mark)

8. Summarise in your own words how the proof arrives at the conclusion that M itself must be monadic.

Main ideas:

No triadic prime can divide M (1 mark).

The proof shows that 2 cannot divide M (1 mark).

The proof uses the sub-proof to show M is monadic (1 mark).

9. Why is the sub-proof that the product of monadic numbers is monadic included in the proof?

So that we can show M itself is a monadic number (1 mark). This provides our contradiction (1 mark).

Do not accept: 'It is just there as extra information, it is not actually needed in the proof' or 'There is no proof, it is just a statement' or 'So we can conclude no triadic prime divides M '.

10. Do lines 3-7, which establish that M is not divisible by a triadic prime, depend on the statements in lines 1-2, which establish that the product of monadic primes is monadic? Explain your answer.

The lines are logically independent (1 mark). Student provides good reasoning (1 mark).

Do not accept: 'Lines 3-7 rely (logically depend) on statements made in lines 1-2' or 'Lines 1-2 rely (logically depend) on statements made in lines 3-7'.

11. Using the method of the proof you have been working with, what would be an appropriate value for M if you were writing the proof for the theorem that there are infinitely many primes of the form $6k + 5$?

$M = 6p_2 \dots p_n + 5$ (1 mark) where $p_1 = 5$ (1 mark).

12. Is the product of two triadic numbers triadic? Why, therefore, would this prevent the methods used in the proof you have been working with from being used to prove there are infinitely many monadic primes?

No it is not (1 mark).

Because the product of triadic numbers need not be triadic, we cannot use triadic numbers in the same way; this proof uses the fact that the product of monadic numbers is monadic (1 mark).

Do not accept: Because the proof uses a contradiction of monadic numbers and since we are trying to prove there are infinitely many monadic numbers, we cannot contradict what we are trying to prove.

13. If 3, 7, 11, 19 were the only triadic primes, what would the value of M be?

$M = 4p_2 \dots p_n = 4 \times 7 \times 11 \times 19 + 3 = 5852 + 3 = 5855$ (2 marks).

14. Could $M = 87$, where M is defined as in this proof, if there were only 2 triadic primes? If yes, state the values of these 2 triadic primes. If no, explain why.

No (1 mark) because $p_1 = 3$ so $87 = 4p_2 + 3$ so $\frac{84}{4} = p_2$ so $21 = p_2$ which is not prime (1 mark).

2 Comprehension Test A' (Experiment 3)

The list below shows the questions used for the multiple choice comprehension test for Proof A in Experiment 3; correct answers are indicated, and question order was randomised for each participant.

1. According to the proof, which of the following would be the first possible value for M ?
 - (a) $M = 87$.
 - (b) $M = 135$.
 - (c) $M = 311$. [correct]

2. In line (L7), why does the proof show that 2 does not divide M ?
 - (a) Because 2 is neither monadic nor triadic but is a prime so it needs to be shown not to divide M for M to be monadic. [correct]
 - (b) Because 2 can also be considered as a triadic prime so for M to be monadic we must show that all triadic primes do not divide M .
 - (c) Because 2 is the only even prime number so if 2 does not divide M then no even number will divide M .

3. Which of the following best defines a prime number?
 - (a) Any real number that greater than 0 and is only divisible by 1 and itself.
 - (b) Any positive integer that is only divisible by 1 and itself.
 - (c) Any positive integer that is greater than 1 that is only divisible by 1 and itself. [correct]

4. Which of the following best describes the logical relation between lines (L2) and (L8)?
 - (a) The lines are logically independent.
 - (b) (L2) logically depends on statements made in line (L8).
 - (c) (L8) logically depends on statements made in line (L2). [correct]

5. Which of the following best describes the logical relation between lines (L5) and (L6)?
 - (a) The lines are logically independent.
 - (b) (L5) logically depends on statements made in line (L6).
 - (c) (L6) logically depends on statements made in line (L5). [correct]

6. Using the method of the proof you have been working with, which of the following would be an appropriate M to use if you were trying to prove there were infinitely many primes of the form $6k + 5$?
- (a) $M = 4p_2 \dots p_n + 5$ where $p_1 = 5$.
 - (b) $M = 6p_2 \dots p_n + 5$ where $p_1 = 6$.
 - (c) $M = 6p_2 \dots p_n + 5$ where $p_1 = 5$. [correct]
7. What type of proof is this?
- (a) Proof by contradiction. [correct]
 - (b) Proof by contraposition.
 - (c) Proof by induction.
8. Which of the following summaries best capture the ideas of the proof?
- (a) The proof assumes there are infinitely many triadic primes and uses them to construct a triadic number M that has only monadic prime factors, which would imply M is also monadic. M cannot be monadic as M is triadic. [correct]
 - (b) The proof lets $M = 4p_2 \dots p_n + 3$, where p_i are prime numbers and p_i does not equal 3. Thus, 2 does not divide M because M is odd. Further, p_i does not divide M because it leaves a remainder of 3.
 - (c) The proof introduces monadic primes to be used later on in the proof. It lets $M = 4p_2 \dots p_n + 3$ and shows 2 does not divide M , since 2 is even and M is odd. However, this would not itself create an infinite triadic prime so the proof uses monadic primes to create an infinite triadic prime.
9. Can we conclude from this proof that the product of two triadic primes is itself triadic?
- (a) No - the proof only shows the product of two monadic numbers is monadic. [correct]
 - (b) Yes - triadic and monadic primes are closely linked, as shown in the proof, so we are allowed to assume that the product of two triadic primes is triadic.
 - (c) Yes - this is used in the proof because M is a triadic number and this can only occur if the product of triadic primes is triadic.

10. Why does the proof include the sub-proof that the product of monadic numbers is monadic?

- (a) Because in line (L4) we have a product of monadic number so M itself needs to be shown as monadic.
- (b) Because by showing that the product of monadic numbers is monadic we can then assume the product of triadic numbers is triadic.
- (c) Because the proof uses it in line (L8) to show that M is in fact monadic leading to a contradiction. [correct]

3 Comprehension Test B (Experiments 2 and 3)

The list below shows the questions used for the multiple choice comprehension test for Proof B in Experiment 2; correct answers are indicated, and question order was randomised for each participant.

1. According to the theorem, which of the following is the most appropriate definition of n ?
 - (a) n belongs to the integers.
 - (b) n belongs to the positive integers. [correct]
 - (c) n belongs to the negative integers.

2. Which justification best explains why $3n^2 + 8 = 2(6m^2 + 4)$ implies $3n^2 + 8$ is even?
 - (a) $6m^2 + 4$ is even because of the plus 4, so $3n^2 + 8$ must be even.
 - (b) $6m^2 + 4$ is even so $2(6m^2 + 4)$ is also even.
 - (c) $6m^2 + 4$ is just another integer, say k , so by definition $3n^2 + 8$ is even because $3n^2 + 8 = 2k$. [correct]

3. Which justification best explains why showing that if n is odd then $3n^2 + 8$ is odd helps to prove the theorem?
 - (a) Because all numbers are either odd or even so if $3n^2 + 8$ is odd because n is odd, then if $3n^2 + 8$ is even, n has to be even.
 - (b) Because we showed in the first half of the proof that if n is even then $3n^2 + 8$ is even. Therefore, by showing if n is odd then $3n^2 + 8$ is odd, we can conclude that n is even if and only if $3n^2 + 8$ is even. [correct]
 - (c) We have shown this in the first half of the proof. The second half is a proof by contradiction which adds to the proof.

4. Which of the following best describes the logical relation between lines (L2) and (L6)?
 - (a) The lines are logically independent. [correct]
 - (b) (L2) logically depends on statements made in line (L6).
 - (c) (L6) logically depends on statements made in line (L2).

5. Which of the following best describes the logical relation between lines (L5), (L6) and (L8)?
- (a) The lines are logically independent.
 - (b) (L8) logically depends on statements made in both lines (L5) and (L6). [correct]
 - (c) (L8) logically depends on statements made in line (L5) and is independent to statements made in line (L6).
6. Which of the following best explains why the proof does not stop at line (L4)?
- (a) Because the proof would be incomplete - we need to show both implications of the if and only if statement. [correct]
 - (b) Because the proof would be incomplete - we need to show if $3n^2 + 8$ is odd then n is odd also.
 - (c) The proof is complete at this point but the extra lines are additional pieces of information to help understanding.
7. Which of the following best describes the method of the second half of this proof?
- (a) Proof by contradiction.
 - (b) Proof by contraposition. [correct]
 - (c) Proof by induction.
8. Which of the following best summarises the proof after line (L5)?
- (a) We assume n is odd, so $n = 2a + 1$. We replace n with $2a + 1$, re-arrange the terms and factorise. This gives $2(6a^2 + 6a + 5) + 1$. This is odd because of the plus 1. Therefore, by contradiction, n is even if and only if $3n^2$ is even.
 - (b) We assume n is odd, so $n = 2a + 1$. We replace n with $2a + 1$, re-arrange the terms and factorise. This gives $2(6a^2 + 6a + 5) + 1$. This is odd because we can replace $6a^2 + 6a + 5$ with, say k , as $6a^2 + 6a + 5$ is just an integer. Therefore, $2(6a^2 + 6a + 5) + 1 = 2k + 1$ which shows $3n^2 + 8$ is odd. Hence by contraposition, n is even if $3n^2 + 8$ is even. [correct]
 - (c) We assume n is odd, so $n = 2a + 1$. We replace n with $2a + 1$, re-arrange the terms and factorise. This gives $2(6a^2 + 6a + 5) + 1$. This is odd because $6a^2 + 6a + 5$ is just an integer. Therefore, $2(6a^2 + 6a + 5) + 1 = 2k + 1$ which shows $3n^2 + 8$ is odd. But this is a contradiction as we assumed n was even in the first half of the proof. Hence by contradiction, n is even if and only if $3n^2 + 8$ is even.

9. Which of the following best describes why we do not explicitly prove $n > 0$ and $n \in \mathbb{Z}$?
- (a) Because we are allowed to choose an arbitrary $n > 0$ where $n \in \mathbb{Z}$. [correct]
 - (b) Because we are proving n is even if and only if $3n^2 + 8$ is even and not $n > 0$ where $n \in \mathbb{Z}$.
 - (c) Because it is clear $n > 0$ where $n \in \mathbb{Z}$. Working with $n > 0$ where $n \in \mathbb{R}$, say, would require a more complex proof.
10. According to the theorem, if $3n^2 + 8$ was an odd number, would this imply that n was odd also?
- (a) No as the theorem talks about even numbers, not odd numbers.
 - (b) Yes because this is the contrapositive of the theorem statement. [correct]
 - (c) Yes, but not because of the theorem - it is because any odd number squared, times three and plus eight gives an odd number.

4 Comprehension Test C (Experiment 2)

The list below shows the questions used for the multiple choice comprehension test for Proof C in Experiment 2; correct answers are indicated, and question order was randomised for each participant.

1. Which of the following best defines the symbol \equiv in this proof?
 - (a) Equivalent to.
 - (b) Congruent to. [correct]
 - (c) Equal to.

2. Which justification best explains why p cannot be 2?
 - (a) Because 2 divides into 4 so you cannot have $p \equiv 2(\text{mod } 4)$.
 - (b) Because $2(\text{mod } 4) = (-1)(\text{mod } p)$ which is shown later in the proof.
 - (c) Because $4n^2 + 1$ is odd so 2 does not divide into it. [correct]

3. Which justification best explains why $y^2 + 1 \equiv 0$?
 - (a) Because $y^2 + 1$ is divisible by n .
 - (b) Because $y^2 + 1 = (2n)^2 + 1 = 4n^2 + 1$. [correct]
 - (c) Because p does not divide n so $y^2 + 1 \equiv 0(\text{mod } p)$.

4. Which of the following best describes the logical relation between lines (L1) and (L2)?
 - (a) The lines are logically independent.
 - (b) (L1) logically depends on statements made in line (L2). [correct]
 - (c) (L2) logically depends on statements made in line (L1).

5. Which of the following best describes the logical relation between lines (L4), (L5) and (L6)?
 - (a) The lines are logically independent.
 - (b) (L6) logically depends on statements made in both lines (L4) and (L5). [correct]
 - (c) (L6) logically depends on statements made in line (L5) and is independent to statements made in line (L4).

6. Which of the following best explains why showing that $p \not\equiv 3 \pmod{4}$ proves the theorem?
- (a) 3 is the first odd prime number. Therefore, if $p \not\equiv 3 \pmod{4}$, it had to be $1 \pmod{4}$ because primes are only divisible by themselves and 1.
 - (b) Prime numbers are either monadic ($1 \pmod{4}$), triadic ($3 \pmod{4}$) or 2. Since we are told that p cannot be 2, by showing it cannot be triadic it has to be monadic. [correct]
 - (c) $3 \pmod{4} = (-1) \pmod{4}$. Therefore, if $p \not\equiv 3 \pmod{4}$, $p \not\equiv (-1) \pmod{4}$. This means it must be $1 \pmod{4}$ by rules of modulo arithmetic.
7. Which of the following best describes the method of this proof?
- (a) Proof by contradiction. [correct]
 - (b) Proof by contraposition.
 - (c) Proof by example.
8. Which of the following best summarises the proof after line (L3)?
- (a) We are told $y^2 + 1 \equiv 0 \pmod{p}$. By doing some substitutions we show $y^{p-1} \equiv (-1) \pmod{p}$. But this cannot be the case because we know p divides $4n^2 + 1$ and by Fermat's Little Theorem, $y^{p-1} \equiv 1 \pmod{p}$. Therefore, we have shown $p \not\equiv 3 \pmod{4}$ and proved the theorem. [correct]
 - (b) We are told $y^2 + 1 \equiv 0 \pmod{p}$. We show $y^{p-1} \equiv (-1) \pmod{p}$ by doing some substitutions. But this cannot be the case because we know $y^2 + 1 \equiv 0 \pmod{4}$ so $y \not\equiv (-1) \pmod{p}$. Therefore, we have proven $p \not\equiv 3 \pmod{4}$ and proved the theorem.
 - (c) We are told $y^2 + 1 \equiv 0 \pmod{p}$. We show $y^{p-1} \equiv (-1) \pmod{p}$ by using Fermat's Little Theorem. But this cannot be the case because we know $p \neq 2$ and y^{4k+2} divides 2. Therefore, we have proven $p \not\equiv 3 \pmod{4}$ and proved the theorem.

9. Which of the following best explains why we set $y = 2n$?
- (a) Because we know $p \neq 2$ so if $y = 2n$, p divides y which cannot be the case. Therefore, setting $y = 2n$ helps us to prove $p \not\equiv 3 \pmod{4}$.
 - (b) Because we can use Fermat's Little Theorem to show $y^{p-1} \equiv 1 \pmod{p}$. This is then used to show $p \not\equiv 3 \pmod{4}$ because by modulo arithmetic, $y^{p-1} \equiv 1 \pmod{p}$ implies $p \equiv 1 \pmod{4}$.
 - (c) Because we can use Fermat's Little Theorem to show $y^{p-1} \equiv 1 \pmod{p}$ and because $y^2 + 1 = 4n^2 + 1$, which is divisible by p by the theorem. This then sets up a contradiction which we use to prove $p \not\equiv 3 \pmod{4}$. [correct]
10. According to the theorem, is $133 \equiv 1 \pmod{4}$?
- (a) Yes because $p = 133$ divided by 4 is 33.25 which is $1 \pmod{4}$.
 - (b) No because $p = 133$ is not prime. [correct]
 - (c) No because $p = 133$ does not divide $(4n^2 + 1) \forall n \in \mathbb{Z}$.

5 Replacement Materials for Experiment 3

Experiment 3 involved change in the example proof and practice proofs provided as part of the self-explanation training; this also necessitated minor changes in the commentary giving illustrative self-explanations and discussing how to avoid simply monitoring or paraphrasing. The new text appears below, in the order in which it appeared in the training: example proof, commentary, practice proof. Other text in the training was left unchanged.

5.1 Replacement Example Proof and Commentary

Theorem:

No odd integer can be expressed as the sum of three even integers.

Proof:

(L1) Assume, to the contrary, that there is an odd integer x , such that $x = a + b + c$, where a, b , and c are even integers.

(L2) Then $a = 2k, b = 2l$, and $c = 2p$, for some k, l, p integers.

(L3) Thus $x = a + b + c = 2k + 2l + 2p = 2(k + l + p)$.

(L4) It follows that x is even; a contradiction.

(L5) Thus no odd integer can be expressed as the sum of three integers. □

After reading this proof, one student made the following self-explanations:

- This proof uses the technique of proof by contradiction.
- Since a, b and c are even integers, we have to use the definition of an even integer, which is used in line 2.
- The proof then replaces a, b and c with their respective definitions in the formula for x .
- The formula for x is then simplified and is shown to satisfy the definition of an even integer also; a contradiction.
- Therefore, no odd integer can be expressed as the sum of three even integers.

You must also be aware that the self-explanation strategy is not the same as monitoring or paraphrasing. These two methods will not help your learning to the same extent as self-explanation.

Paraphrasing

“ a, b and c have to be positive or negative, even whole numbers”

There is no self-explanation in this statement. No additional information is added or linked. The student merely uses different words to describe what is already represented in the text by the words “even integers”. You should avoid using such paraphrasing during your own text comprehension. Paraphrasing will not help your understanding of the text as much as self-explanation will.

Monitoring

“OK, I understand that $2(k + l + p)$ is an even integer.”

This statement simply shows the student’s thought process. It is not the same as self-explanation where the student relates the sentence to additional information in the text or prior knowledge. Please concentrate on self-explanation rather than monitoring.

A possible self-explanation of the same sentence would be:

“OK, $2(k + l + p)$ is an even integer because the sum of 3 integers is an integer and 2 times an integer is an even integer.”

In this example the student identifies and elaborates the main ideas in the text. They use information that has already been presented to them to help with their understanding of how the proof is logically connected. This is the approach you should take after reading every line of a proof in order to improve your understanding of the material.

5.2 Replacement Practice Proof

Theorem:

There is no smallest positive real number.

Proof:

(L1) Assume, to the contrary, that there exists a smallest positive real number.

(L2) Therefore, by assumption, there exists a real number r such that $0 < r < s$ where s is any other positive real number.

(L3) Consider $m = \frac{r}{2}$.

(L4) Clearly, $0 < m < r$.

(L5) Therefore, this is a contradiction since m is a positive real number that is smaller than r .

(L6) Thus there is no smallest positive real number. □